PRELIMINARY EXAM IN ANALYSIS FALL 2016

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) (a) State the three convergence theorems of Lebesgue integration theory: (1) the monotone convergence theorem; (2) Fatou's lemma; (3) the dominated convergence theorem.

(b) Use Fatou's lemma to prove the dominated convergence theorem.

(2) Let f be a measurable function on a measure space $(X, \mathscr{F}, \mu)$. Let $0 \le p < q < \infty$ and assume

$$\int_X |f|^p \, d\mu < \infty \quad \text{and} \quad \int_X |f|^q \, d\mu < \infty.$$

Show that

$$\int_X |f|^r \, d\mu < \infty$$

for all $r \in [p, q]$.

(3) Let $(X, \mathscr{F}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions such that $f_n \to f$ almost everywhere. Suppose that there are constants $p > 1$ and C such that $\|f_n\|_p \le C$ for all n. Show that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

(Hint: Truncate at a large number K and let $K \to \infty$.)

(4) Let $\{f_n\}$ be a sequence of nondecreasing functions on a finite interval $[a, b]$ such that $f_n(a) = 0$ for all n and

$$\sum_{n=1}^{\infty} f_n(b) < \infty.$$

Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Show that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

for almost every $x \in [a, b]$. You may use without proof the fact that every monotone function on the real line $\mathbb{R}$ is almost everywhere differentiable.

(5) Let f and g be two integrable functions on $I = [0,1]$ such that

$$\int_I g(y)\, dy = 0.$$

Show that for all $p \geq 1$,

$$\int_I |f(x)|^p\, dx \leq \int_{I\times I} |f(x) + g(y)|^p\, dxdy.$$

Note: dx and $dxdy$ denote the Lebesgue measures on I and $I \times I$, respectively.

Part II. Functional Analysis

Do **three** of the following five problems.

(1) (a) Define the convolution $f * g$ of two functions $f, g \in L^1(\mathbb{R})$. Show that $f * g \in L^1(\mathbb{R})$.

(b) For $f, g \in L^1(\mathbb{R})$, calculate the Fourier transform $\widehat{f * g}$. Justify your answer.

(c) Define the convolution $f * g$ of two functions $f, g \in L^2(\mathbb{R})$. Show that

$$|f * g(x)| \leq ||f||_2 ||g||_2$$

and that $f * g \in C_b(\mathbb{R})$ (bounded continuous functions).

(2) Suppose $f \in L^1(\mathbb{R}), f \geq 0, f \neq 0$. Show that the Fourier transform $\hat{f}$ is in $C_b(\mathbb{R})$ (the bounded continuous functions) and that $\sup |\hat{f}(\xi)|$ is obtained at $\xi = 0$ and only at $\xi = 0$.

(3) Does the Fourier series of every $f \in C(S^1)$ converge pointwise? Here $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the unit circle. Prove that your answer is correct.

(Hint: Recall that the partial sums $S_N(f) = D_N * f$ where D_N is the Dirichlet kernel, $D_N(t) = \frac{\sin(N+\frac{1}{2})t}{\sin \frac{1}{2}t}$. You may use without proof that $||D_N||_{L^1(S^1)} \geq C \log N$.)

(4) Let H be a Hilbert space and $A : H \to H$ a bounded linear operator. Show that if $\langle Au, u\rangle = 0$ for all u, then $A = 0$.

(5) Let $f \in L^2[0,1]$ and define

$$Tf(x) = \int_0^x f(y)dy.$$

(a) Show that T is a compact operator from $L^2[0,1]$ to itself.

(b) Compute the adjoint of T. Is T self-adjoint on $L^2[0,1]$?

(c) Find all eigenvalues of T. Show that for $z \neq 0$ the operator $(T - z\,I)$ is 1-1 and onto, where I is the identity operator.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function.

  (a) Show that if $\operatorname{Re} f$ is bounded from above then f is constant.

  (b) Show that if $f(z)$ is real when $|z| = 1$ then f is constant.

(2) In each of the following cases, find a conformal mapping from the domain Ω onto the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$.

  (a) $\Omega = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < \pi\}$.

  (b) $\Omega = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0,\ |z| < 1\}$.

(3) Suppose that f is analytic in a neighborhood of the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and has power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that if f has exactly m zeros (counted with multiplicity) inside $\mathbb{D}$ then

$$\inf_{|z|=1} |f(z)| \leq |a_0| + |a_1| + \cdots + |a_m|.$$

(4) (a) For $p, q > 0$ with $p \neq q$, show using Residue Theory that

$$\int_0^{\infty} \frac{dx}{(x^2 + p^2)(x^2 + q^2)} = \frac{\pi}{2pq(p+q)}.$$

  (b) What is the integral if $p = q$?

(5) Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function, where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the unit disc. Suppose that $f(0) = f'(0) = 0$. Show that $|f''(0)| \leq 2$ with $|f''(0)| = 2$ if and only if $f(z) = \alpha z^2$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

(Hint: first show that $f(z) = zg(z)$ for $g : \mathbb{D} \to \mathbb{D}$ analytic with $g(0) = 0$.)